

# GORENSTEIN TORIC RINGS OF COMPRESSED CUT POLYTOPES

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**ABSTRACT.** An integral convex polytope is said to be compressed if all of whose pulling triangulations (reverse lexicographic triangulations) are unimodular. It is known that the cut polytope of a graph  $G$  is compressed if and only if  $G$  has no  $K_5$ -minor and no induced cycle of length  $\geq 5$ . In this paper, compressed cut polytopes of graphs whose toric ring is Gorenstein are studied. We show that, if the cut polytope of a graph  $G$  is compressed, then its toric ring is Gorenstein if and only if either  $G$  is a bipartite graph or a chordal graph without bridges.

## INTRODUCTION

Let  $G = (V, E)$  be a finite graph on the vertex set  $V = [n] = \{1, 2, \dots, n\}$  and the edge set  $E = \{e_1, \dots, e_m\}$ . We assume that  $G$  has no loops and no multiple edges. Given  $S \subset [n]$ , the *cut semimetric* on  $G$  induced by  $S$  is the 0/1 vector  $\delta_G(S) = (d_{ij} \mid \{i, j\} \in E) \in \mathbb{R}^m$  where

$$d_{ij} := \begin{cases} 1 & \text{if } |S \cap \{i, j\}| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for each  $\{i, j\} \in E$ . In particular, we have  $\delta_G(\emptyset) = (0, \dots, 0) \in \mathbb{R}^m$ . The *cut polytope*  $\text{Cut}^\square(G)$  of  $G$  is the convex hull of  $\{\delta_G(S) \mid S \subset [n]\} \subset \mathbb{Z}^m$ . If  $S' = [n] \setminus S$ , then we have  $\delta_G(S) = \delta_G(S')$ . It then follows that  $\text{Cut}^\square(G)$  has  $2^{n-1}$  vertices. As explained in [3], cut polytopes are well-known and important objects in discrete mathematics (graph theory, combinatorial optimization, and so on).

Let  $K[\mathbf{x}, t] = K[x_1, \dots, x_m, t]$  denote the polynomial ring in  $m + 1$  variables over a field  $K$ . The *toric ring* of the cut polytope  $\mathcal{P} = \text{Cut}^\square(G)$  is the subalgebra  $K[\mathcal{P}]$  of  $K[\mathbf{x}, t]$  generated by those squarefree monomials  $\mathbf{x}^{\mathbf{a}}t = x_1^{a_1} \cdots x_m^{a_m}t$  such that  $\mathbf{a} = (a_1, \dots, a_m)$  is a vertex of  $\mathcal{P}$ . We regard  $K[\mathcal{P}]$  as a homogeneous algebra by setting each  $\deg \mathbf{x}^{\mathbf{a}}t = 1$  and write  $F(K[\mathcal{P}], \lambda)$  for its Hilbert series. Then,

$$F(K[\mathcal{P}], \lambda) = \frac{h_0 + h_1\lambda + \cdots + h_s\lambda^s}{(1 - \lambda)^{d+1}},$$

where each  $h_i \in \mathbb{Z}$  with  $h_s \neq 0$  and where  $d = \dim \mathcal{P}$ . The sequence  $(h_0, h_1, \dots, h_s)$  is called the *h-vector* of  $K[\mathcal{P}]$ . If the toric ring  $K[\mathcal{P}]$  is normal, then  $K[\mathcal{P}]$  is Cohen–Macaulay. If  $K[\mathcal{P}]$  is Cohen–Macaulay, then the *h-vector* of  $K[\mathcal{P}]$  is nonnegative, i.e., each  $h_i \geq 0$ . Moreover, if  $K[\mathcal{P}]$  is Gorenstein, then the *h-vector* of  $K[\mathcal{P}]$  is

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symmetric, i.e.,  $h_i = h_{s-i}$  for all  $i$ . The cut polytope  $\mathcal{P}$  is said to be *Gorenstein* if  $K[\mathcal{P}]$  is Gorenstein.

Sturmfels–Sullivant [10] studied the toric rings of cut polytopes and their application to algebraic statistics. Especially, they showed that a clique sum of the graphs yields a toric fiber product [12] of the toric ideals of cut polytopes. They also gave several interesting conjectures on properties of the toric ideals of cut polytopes. Inspired by them, many results are known on toric ideals and toric rings of cut polytopes. See, e.g., [5, 6, 7, 8, 13]. However, Sturmfels–Sullivant [10] said that “We do not have a firm conjecture on the structure of those graphs whose cut ideal is Gorenstein.” One of the reason for this difficulty is that Gorensteiness is *not* preserved under taking a clique sums of graphs (i.e., toric fiber product). It is known [7] that, if  $G$  is a tree, then  $\text{Cut}^\square(G)$  is Gorenstein. However, no other result seems to be known for Gorenstein cut polytopes. An integral convex polytope is said to be *compressed* if all of whose pulling triangulations (reverse lexicographic triangulations) are unimodular. It is known [11] that the cut polytope of a graph  $G$  is compressed if and only if  $G$  has no  $K_5$ -minor and no induced cycle of length  $\geq 5$ . The purpose of this paper is to classify compressed cut polytopes whose toric ring is Gorenstein.

The contents of this paper is as follows. In Section 1, together with graph theoretical terminology, we introduce a characterization of compressed cut polytopes given by Sullivant [11]. In Section 2, by using theory of special simplices, we show that, if the cut polytope  $\text{Cut}^\square(G)$  of  $G$  is compressed, then its toric ring is Gorenstein if and only if either  $G$  is a bipartite graph or a chordal graph without bridges.

## 1. COMPRESSED CUT POLYTOPES

In this section, we introduce the characterization of compressed cut polytopes studied in [11] and [10, Theorem 1.3] and give equations of facets of compressed cut polytopes.

First, we introduce graph theoretical terminology. Let  $G$  be a graph with the vertex set  $V(G) = [n] = \{1, 2, \dots, n\}$  and the edge set  $E(G)$ . We assume that  $G$  has no loops and no multiple edges. A *cycle* of length  $q$  ( $\geq 3$ ) of  $G$  is a finite sequence of the form

$$(1) \quad \Gamma = (\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_q, v_1\})$$

with each  $\{v_k, v_{k+1}\}$  and  $\{v_q, v_1\}$  belong to  $E(G)$  and  $v_i \neq v_j$  for all  $1 \leq i < j \leq q$ . An *even* (resp. *odd*) *cycle* is a cycle of even (resp. odd) length. A *triangle* is a cycle of length 3. A *chord* of a cycle (1) is an edge  $e \in E(G)$  of the form  $e = \{v_i, v_j\}$  for some  $1 \leq i < j \leq q$  with  $e \notin E(C)$ . An *induced* cycle of  $G$  is a cycle having no chords. Let  $e = \{i, j\} \in E(G)$  be an edge of  $G$ . Then, the new graph  $G \setminus e$  on the edge set  $V(G)$  and the edge set  $E(G) \setminus \{e\}$  is called the graph obtained from  $G$  by *deleting* the edge  $e$ . On the other hand, the new graph  $G/e$  obtained by the procedure

- (i) Identify the vertices  $i$  and  $j$ ;
- (ii) Delete the multiple edges that may be created while (i);

is called the graph obtained from  $G$  by *contracting* the edge  $e$ . A graph  $H$  is said to be a *minor* of  $G$  if it can be obtained from  $G$  by a sequence of deletions and/or contractions of edges (and deletions of vertices).

Let  $K_n$  denote the complete graph with  $n$  vertices. Then the following characterization is known ([11] and [10, Theorem 1.3]):

**Proposition 1.1.** *Let  $G$  be a graph. Then, the following conditions are equivalent:*

- (i)  $\text{Cut}^\square(G)$  is compressed;
- (ii)  $\text{Cut}^\square(G)$  has a reverse lexicographic unimodular triangulation;
- (iii)  $G$  has no  $K_5$ -minor and no induced cycle of length  $\geq 5$ .

In addition, by Barahona–Mahjoub’s formula [2] together with Proposition 1.1, we have the following:

**Proposition 1.2.** *Suppose that the cut polytope  $\text{Cut}^\square(G)$  of a graph  $G$  is compressed. Then,  $\text{Cut}^\square(G)$  is the solution set of the following linear inequalities:*

$$\begin{aligned} 0 \leq x_i &\leq 1, & (e_i \text{ does not belong to any triangle of } G), \\ x_i - x_j - x_k &\leq 0, & (\{e_i, e_j, e_k\} \text{ is a triangle of } G), \\ x_i + x_j + x_k &\leq 2, & (\{e_i, e_j, e_k\} \text{ is a triangle of } G), \\ 0 \leq x_i + x_j + x_k - x_\ell &\leq 2, & (\{e_i, e_j, e_k, e_\ell\} \text{ is an induced cycle of } G \text{ of length } 4). \end{aligned}$$

Moreover, each of them defines a facet of  $\text{Cut}^\square(G)$ .

## 2. GORENSTEIN CUT POLYTOPES

In this section, by using theory of special simplices, we classify compressed cut polytopes whose toric ring is Gorenstein. In general, Gorensteiness is *not* preserved under the following operations of graphs:

- taking a clique sums of graphs (i.e., toric fiber product [12]),
- taking an edge contraction,
- taking an edge deletion,
- taking an induced subgraph.

In fact,

**Example 2.1.** Let  $K_n$  be the complete graph with  $n$  vertices and let  $G_{m,n}$  be a 0-sum (i.e., glued at a vertex) of  $K_m$  and  $K_n$ . By Proposition 1.1,  $\text{Cut}^\square K_n$  is compressed if and only if  $n \leq 4$ . Moreover,  $\text{Cut}^\square G_{m,n}$  is compressed if and only if  $m, n \leq 4$ .

- (a) Both  $\text{Cut}^\square K_2$  and  $\text{Cut}^\square K_3$  are Gorenstein (simplices). However,  $\text{Cut}^\square G_{2,3}$  is not Gorenstein.
- (b) The cut polytope  $\text{Cut}^\square G_{3,3}$  is Gorenstein. However,  $\text{Cut}^\square G_{2,3}$  is not Gorenstein. Note that  $G_{2,3}$  is an induced subgraph of  $G_{3,3}$ . In addition,  $G_{2,3}$  is obtained by an edge contraction from  $G_{3,3}$ .

It is known [7] that, if  $G$  is a tree, then  $\text{Cut}^\square(G)$  is Gorenstein. However, no other result seems to be known for Gorenstein cut polytopes.

We now apply the theory of special simplices given by Athanasiadis. Let  $\mathcal{P} \subset \mathbb{R}^m$  be a convex polytope. A  $d$ -simplex  $\Sigma$  each of whose vertices is a vertex of  $\mathcal{P}$  is called

a *special simplex* in  $\mathcal{P}$  if each facet of  $\mathcal{P}$  contains exactly  $d$  of the vertices of  $\Sigma$ . It is known ([1, 9]) that

**Proposition 2.2.** *Let  $\mathcal{P}$  be a compressed polytope. Then there exists a special simplex in  $\mathcal{P}$  if and only if its toric ring  $K[\mathcal{P}]$  is Gorenstein. Moreover, if there exists a special simplex in  $\mathcal{P}$ , then  $h$ -vector of  $\mathcal{P}$  is unimodal.*

From now on, we discuss the existence of special simplices.

**Lemma 2.3.** *Suppose that  $\text{Cut}^\square(G)$  is compressed and possesses a special simplex  $\Sigma$  of  $\text{Cut}^\square(G)$ . If  $G$  satisfies one of the following conditions, then we have  $\dim \Sigma = 1$ :*

- (a) *There exists an edge  $e_i$  of  $G$  such that no triangle of  $G$  contains  $e_i$ ;*
- (b)  *$G$  has an induced cycle of length 4.*

*Proof.* Suppose that there exists an edge  $e_i$  of  $G$  such that no triangle of  $G$  contains  $e_i$ . By Proposition 1.2, the inequality  $0 \leq x_i \leq 1$  defines two facets  $\mathcal{F}$  and  $\mathcal{F}'$  of  $\text{Cut}^\square(G)$ . Note that each vertex of  $\text{Cut}^\square(G)$  belongs to exactly one of  $\mathcal{F}$  and  $\mathcal{F}'$ . Thus, the special simplex  $\Sigma$  has exactly two vertices.

Suppose that  $G$  has an induced cycle of length 4. By Proposition 1.2, the inequality  $0 \leq x_i + x_j + x_k - x_\ell \leq 2$  defines two facets  $\mathcal{F}$  and  $\mathcal{F}'$  of  $\text{Cut}^\square(G)$ . Since each vertex of  $\text{Cut}^\square(G)$  belongs to exactly one of  $\mathcal{F}$  and  $\mathcal{F}'$ , the special simplex  $\Sigma$  has exactly two vertices.  $\square$

**Lemma 2.4.** *Suppose that  $\text{Cut}^\square(G)$  is compressed and possesses a special simplex  $\Sigma$  of  $\text{Cut}^\square(G)$ . If  $G$  has a triangle, then we have  $\dim \Sigma = 3$ .*

*Proof.* Let  $C = \{e_i, e_j, e_k\}$  be a triangle of  $G$ . By Proposition 1.2, there are 4 facets of  $\text{Cut}^\square(G)$  arising from  $C$ :

$$\begin{aligned}\mathcal{F}_1 : x_i - x_j - x_k &\leq 0, \\ \mathcal{F}_2 : x_j - x_i - x_k &\leq 0, \\ \mathcal{F}_3 : x_k - x_i - x_j &\leq 0, \\ \mathcal{F}_4 : x_i + x_j + x_k &\leq 2.\end{aligned}$$

For each vertex  $(x_1, \dots, x_m) \in \{0, 1\}^m$  of  $\text{Cut}^\square(G)$ , possible  $(x_i, x_j, x_k)$ 's are  $(0, 0, 0)$ ,  $(0, 1, 1)$ ,  $(1, 0, 1)$  and  $(1, 1, 0)$ . Moreover, it follows that

$$\begin{aligned}(x_1, \dots, x_m) \notin \mathcal{F}_1 &\Leftrightarrow (x_i, x_j, x_k) = (0, 1, 1), \\ (x_1, \dots, x_m) \notin \mathcal{F}_2 &\Leftrightarrow (x_i, x_j, x_k) = (1, 0, 1), \\ (x_1, \dots, x_m) \notin \mathcal{F}_3 &\Leftrightarrow (x_i, x_j, x_k) = (1, 1, 0), \\ (x_1, \dots, x_m) \notin \mathcal{F}_4 &\Leftrightarrow (x_i, x_j, x_k) = (0, 0, 0).\end{aligned}$$

Thus, the special simplex  $\Sigma$  has exactly four vertices, as required.  $\square$

A graph  $G$  is called a *bipartite graph* if there exists a bipartition  $V(G) = V_1 \cup V_2$  such that any edge of  $G$  connects a vertex of  $V_1$  and a vertex of  $V_2$ . It is known that  $G$  is bipartite if and only if  $G$  has no odd cycle. A graph  $G$  is said to be *chordal* if  $G$  has no induced cycle of length  $\geq 4$ . An edge  $e$  of  $G$  is called a *bridge* if there exists

no cycle of  $G$  containing  $e$ . It is easy to see that a chordal graph  $G$  has no bridges if and only if

$$E(G) = \bigcup_{C: \text{triangle of } G} E(C).$$

We now come to a main result of the present paper.

**Theorem 2.5.** *Let  $G$  be a graph. Suppose that the cut polytope  $\text{Cut}^\square(G)$  of  $G$  is compressed. Then  $\text{Cut}^\square(G)$  has a special simplex (i.e.,  $\text{Cut}^\square(G)$  is Gorenstein) if and only if  $G$  satisfies one of the following:*

- (i)  $G$  is bipartite;
- (ii)  $G$  is a chordal graph without bridges.

*Proof.* Suppose that  $\text{Cut}^\square(G)$  is compressed.

( $\Rightarrow$ ) Suppose that  $\text{Cut}^\square(G)$  has a special simplex  $\Sigma$ . If  $G$  is not bipartite, then  $G$  has an odd cycle. Since  $G$  has no induced odd cycle of length  $\geq 5$ ,  $G$  has a triangle. By Lemma 2.4, we have  $\dim \Sigma = 3$ . Since  $\dim \Sigma \neq 1$ ,  $G$  satisfies neither (a) nor (b) in Lemma 2.3. Thus,  $G$  is chordal and, for each  $e \in E(G)$ , there exists a triangle  $C$  of  $G$  such that  $e \in E(C)$ .

( $\Leftarrow$ ) Suppose that  $G$  is bipartite. Let  $V(G) = V_1 \cup V_2$  be a bipartition of  $V(G)$  and let  $E(G) = \{e_1, \dots, e_m\}$ . We will show that the simplex  $\Sigma = \text{Conv}(\mathbf{0}, \mathbf{1})$ , where  $\mathbf{0} = (0, \dots, 0)$  and  $\mathbf{1} = (1, \dots, 1)$  is a special simplex. Note that, both  $\delta_G(\emptyset) = \mathbf{0}$  and  $\delta_G(V_1) = \mathbf{1}$  are vertices of  $\text{Cut}^\square(G)$ . By Proposition 1.2, the facets of  $\text{Cut}^\square(G)$  are defined by the following inequalities:

$$\begin{aligned} 0 \leq x_i \leq 1, & \quad (1 \leq i \leq m), \\ 0 \leq x_i + x_j + x_k - x_\ell \leq 2, & \quad (\{e_i, e_j, e_k, e_\ell\} \text{ is a cycle of } G). \end{aligned}$$

For each facet  $\mathcal{F}$  of  $\text{Cut}^\square(G)$ , exactly one of  $\mathbf{0}$  and  $\mathbf{1}$  belongs to  $\mathcal{F}$ . Thus,  $\Sigma$  is a special simplex.

On the other hand, suppose that  $G$  is chordal and, for each  $e \in E(G)$ , there exists a triangle  $C$  of  $G$  such that  $e \in E(C)$ . Since  $G$  is chordal and has no  $K_5$  as a subgraph,  $G$  is four-colorable (see, e.g., [4]). Let  $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$  be a four coloring of  $G$ . We define the vertex  $\mathbf{u}_i$  of  $\text{Cut}^\square(G)$  by  $\mathbf{u}_i = \delta_G(V_i)$  for each  $i = 1, 2, 3, 4$ . We will show that  $\Sigma = \text{Conv}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$  is a special simplex of  $\text{Cut}^\square(G)$ . By Proposition 1.2, the facets of  $\text{Cut}^\square(G)$  are defined by the following inequalities:

$$\begin{aligned} x_i - x_j - x_k \leq 0, & \quad (\{e_i, e_j, e_k\} \text{ is a triangle of } G), \\ x_i + x_j + x_k \leq 2, & \quad (\{e_i, e_j, e_k\} \text{ is a triangle of } G). \end{aligned}$$

Let  $C = \{e_i, e_j, e_k\}$  be a triangle of  $G$  where  $e_i = \{s, t\}$ ,  $e_j = \{t, u\}$  and  $e_k = \{s, u\}$ . There are 4 facets arising from  $C$ :

$$\begin{aligned} \mathcal{F}_1 : x_i - x_j - x_k &\leq 0, \\ \mathcal{F}_2 : x_j - x_i - x_k &\leq 0, \\ \mathcal{F}_3 : x_k - x_i - x_j &\leq 0, \\ \mathcal{F}_4 : x_i + x_j + x_k &\leq 2. \end{aligned}$$

Without loss of generality, we may assume that  $s \in V_2$ ,  $t \in V_3$  and  $u \in V_4$ . Then,

$$\begin{aligned} \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 &\in \mathcal{F}_1, \mathbf{u}_4 \notin \mathcal{F}_1, \\ \mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_4 &\in \mathcal{F}_2, \mathbf{u}_2 \notin \mathcal{F}_2, \\ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4 &\in \mathcal{F}_3, \mathbf{u}_3 \notin \mathcal{F}_3, \\ \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 &\in \mathcal{F}_4, \mathbf{u}_1 \notin \mathcal{F}_4. \end{aligned}$$

This discussion is independent of the choice of a triangle of  $G$ . Thus,  $\Sigma$  is a special simplex, as desired.  $\square$

**Example 2.6.** The following graphs satisfy the condition in Theorem 2.5:

- Trees;
- $2 \times m$  bipartite graphs;
- 2-connected chordal graphs having no  $K_5$  as a subgraph.

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